MAR IVANIOS COLLEGE (AUTONOMOUS)
THIRUVANANTHAPURAM

Reg. No. :………………… **Name :…………………..**

Fifth Semester B.Sc. Degree Examination, November 2016

First Degree Programme under CBCSS

Core Course: Mathematics – VIII

AUMM545: Abstract Algebra I

Time: **3** Hours Max. Marks: **80**

SECTION – A

*Answer **ALL** questions / problems in one or two sentences.*

1. Are the binary structures $(\mathbb{Q}, +)$ and $(\mathbb{R}, +)$ isomorphic, Justify ?
2. On $\mathbb{Q}$ determine whether the binary operation $*$ given by $a*b=a-b$ is associative.
3. Is $\mathbb{R}^*$ set of non zero real numbers together with a binary operation $*$ defined by $a * b = {}^{a}/_{b}$ a group? Justify your answer.
4. Find the order of the cyclic subgroup of $\mathbb{Z}_4$ generated by 3.
5. Give an example of a finite group that is not cyclic.
6. Express $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 2 & 6 & 3 & 7 & 4 & 5 & 1 \end{pmatrix}$ as a product of transpositions.
7. Find all orbits of the permutation $\sigma: \mathbb{Z} \to \mathbb{Z}$ where $\sigma(n) = n + 2$.
8. Find the index of $< 2 >$ in Z_{12}.
9. According to division algorithm find the remainder *r* when -38 is divided by 7.
10. Find the order of (8, 4, 10) in the group $Z_{12} \times Z_{60} \times Z_{24}$.

(10 × 1 = 10 Marks)

SECTION – B

*Answer any **EIGHT** questions / problems, not exceeding a paragraph.*

11. Prove that if $*$ is an associative and commutative binary operation on a set *S*, then $(a * \text{b}) * (c * d) = [(d * c) * a] * b$ for all $a, b, c, d \in S$, assuming the associative law only for triples.

12. If $\phi: S \to S'$ is an isomorphism of $(S,*)$ with $(S',*')$ and e is an identity element S then prove that $\phi(e)$ is the identity element of for the binary operation $*'$ on S'.
13. Show that Q^+ with the operation $*$ defined by $a * b = \frac{ab}{2}$ is a group.
14. Prove that the identity element and inverse elements are unique in a group.
15. Prove that if G is an abelian group, written multiplicative, with identity element e, then all elements x of G satisfying the equation $x^2 = e$ form a sub group H of G.
16. Show that the collection of permutations S_A of a non – empty set A is a group under permutation multiplication.
17. Show that every permutation σ of a finite set is a product of disjoint cycles.
18. If $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 4 & 5 & 6 & 2 \end{pmatrix}$ and $\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 1 & 3 & 6 & 5 \end{pmatrix}$ in S_6 compute $\sigma^{-1}\tau\sigma$.
19. Find the product of cycles (1,2)(4, 7, 8)(2,1)(7,2,8,1,5) that are permutations of {1, 2, 3, 4, 5, 6, 7, 8}.
20. Let H be a subgroup of G. Prove that the relation $\sim_R$ defined on G by $a \sim_R b$ if and only if $ab^{-1} \in H$ is an equivalence relation on G.
21. Prove that every cyclic group is abelian.
22. Find all cosets of the subgroup of $< 4 >$ of Z_{12}.

(8 × 2 = 16 Marks)

SECTION – C

Short essay type problems: Answer any ***SIX*** *questions.*

23. Define isomorphism of binary structures. Show that the binary structure $(\mathbb{R}, +)$ is isomorphic to the structure $(\mathbb{R}^+, .)$ where the operations are respectively usual addition and multiplication.
24. Show that a non – empty subset H of a group G is a subgroup of G if and only if $ab^{-1} \in H$ for all $a, b \in H$.
25. Show that if G is a group with binary operation *, and if $a, b \in G$, then the linear equations $a * x = b$ and $y * a = b$ have unique solutions $x, y \in G$.
26. Prove that for any group G, $H_G = \{x \in G | xs = sx \text{ for all } s \in G\}$ is an abelian subgroup of G.

27. Show that the sub group of a cyclic group is cyclic.

28. Prove that no permutation in S_n can be expressed both as a product of an even number of transpositions and as a product of an odd number of transpositions.

29. If n ≥ 2, prove that the collection of all even permutations of {1, 2, ..., n} forms a subgroup of the symmetric group S_n of order $\frac{n!}{2}$.

30. State and prove Lagrange's Theorem. Deduce that every group of prime order is cyclic.

31. Suppose H and K are subgroups of a group G such that $K \leq H \leq G$ and suppose $(H: K)$ and $(G: H)$ are both finite. Prove that $(G: K)$ is finite and $(G: K) = (G: H)(H: K)$.

(6 × 4 = 24 Marks)

SECTION – D

Long essay type problems: Answer any ***TWO*** *questions.*

32. a) Define the permutation group S_3. Find all subgroups of S_3 and draw the subgroup diagram.

    b) Let G be a cyclic group. Show that if the order of G is infinite then G is isomorphic to (Z, +) and if G has a finite order n, then G is isomorphic to $(Z_n, +_n)$.

33. a) Let *G* be a cyclic group with *n* elements and generated by *a*. Let $b \in G$ and let $b = a^s$. Then *b* generates a cyclic subgroup *H* of *G* containing *n/d* elements, where *d* is the greatest common divisor of *n* and *s*.

    b) Show that if a is a generator of a finite cyclic group G of order n, then the other generators of G are the elements of the form a^r, where *r* is relatively prime to *n*.

34. a) Let *G* and G' be groups and let $\phi: G \to G'$ be a one-to-one function such that $\phi(xy) = \phi(x)\phi(y)$ for all $x, y \in G$. Then prove that $\phi(G)$ is a subgroup of G' and ϕ provides an isomorphism of G with $\phi(G)$.

    b) State and prove Cayley's theorem.

35. a) Let $G_{1,} G_{2,} \dots, G_n$ be groups. For $(a_1, a_2, \dots, a_n)$ and $(b_1, b_2, \dots, b_n)$ in $\prod_{i=1}^{n} G_i$, define $(a_1, a_2, \dots, a_n)(b_1, b_2, \dots, b_n)$ = $(a_1 b_1, a_2 b_2, \dots, a_n b_n)$. Then prove that $\prod_{i=1}^{n} G_i$ is a group under this binary operation.

b) Prove that $Z_m \times Z_n$ is cyclic and is isomorphic to Z_{mn} if and only if g.c.d(m, n) = 1.

(2 × 15 = 30 Marks)

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